数理解析研究所講究録
431巻 1981年 1-14

A Radon transform in monogenic function theory .

Franciscus Sommen (*)

State University of Ghent

Abstract. In this paper we study the multiple Taylor series expansion of a monogenic function, by making use of a Radon transform in monogenic function theory.

Introduction. In (3), Hayman proved that every harmonic function (and hence every holomorphic function) in $\{z \in \mathbb{C} \mid |z| < R\}$, $z = x + iy$, admits a multiple Taylor series expansion which converges absolutely in the domain which is given by

$$\{(x, y) \in \mathbb{R}^2 \mid |x| + |y| < R\}.$$

In the several dimensional case Siciak studied this problem in (5), by making use of the complex extension of a harmonic function in the unit ball.

In this paper we generalize the result of Hayman for holomorphic functions to the theory of monogenic functions; a theory which has been studied by Delanghe and Brackx in (2) and which is a generalization of the theory of holomorphic functions to several dimensions.

We shall prove that every analytic function, of which the multiple Taylor series converges in the interval

$]-R_1, R_1[\times \ldots \times]-R_m, R_m[$, $R_1 > 0, \ldots, R_m > 0$,

admits a unique left monogenic extension to the domain

$$\{u \in \mathbb{R}^{m+1} \mid |u_0| + |u_1| < R_1, \ldots, |u_0| + |u_m| < R_m\}.$$

Furthermore the multiple Taylor series of the extension converges absolutely in this domain, and this domain is optimal with respect to this type of convergence.

(*) Aspirant N.F.W.O. Belgium

To this end, we construct a generalization:

$P(u, \vec{z}), (u, \vec{z}) \in \mathbb{R}^{m+1} \times \mathbb{C}^{m}$

of the function

$(1 - uz)^{-1}, (u, z) \in \mathbb{C} \times \mathbb{C}$,

which is holomorphic in $\vec{z}$ and monogenic in u, and which is used to generalize the classical Radon transform:

$\pi : \mathcal{H}'(\bar{B}(0,1)) \longrightarrow \mathcal{H}(B(0,1))$, which is given by

$\pi(T)(u) = \langle T_z, (1 - uz)^{-1} \rangle$,

to the theory of monogenic functions.

Preliminaries. In the sequel we always work with modules of functions with values in a complex Clifford algebra.

The complex Clifford algebra over $\mathbb{R}^m$ is defined as follows:

$$\mathcal{A} = \Big\{ \sum_{A \subset \{1,\ldots,m\}} a_A e_A \,\|\, a_A \in \mathbb{C} \Big\}$$

where $e_A = e_{\alpha_1} \ldots e_{\alpha_h}$, when $A = \{\alpha_1, \ldots, \alpha_h\}$; $\alpha_1 < \ldots < \alpha_h$,

and $e_\emptyset = e_0 = 1$, $e_{\{k\}} = e_k$, $k = 1, \ldots, m$.

The involution in $\mathcal{A}$ is defined by $\bar{a} = \sum_{A \subset \{1,\ldots,m\}} \bar{a}_A \bar{e}_A$,

$e_A = e_{\alpha_1} \ldots e_{\alpha_h}$, where $\bar{e}_A = \bar{e}_{\alpha_h} \ldots \bar{e}_{\alpha_1}$ and $\bar{e}_{\alpha_j} = -e_{\alpha_j}$.

As $\mathcal{A}$ is isomorphic to $\mathbb{C}^{2^m}$, we may provide $\mathcal{A}$ with the $\mathbb{C}^{2^m}$ - norm.

Hence, $|a| = \Big(\sum_{A \subset \{1,\ldots,m\}} |a_A|^2 \Big)^{1/2}$.

Furthermore it is easy to show that for any $a, b \in \mathcal{A}$,

$|a . b| \leqslant 2^{m/2} |a| . |b|$.

A point $(x_0, \ldots, x_m)$ of $\mathbb{R}^{m+1}$ shall be identified with the Clifford number $x_0 + \vec{x} = x_0 + \sum_{j=1}^{m} x_j e_j$.

In this way $\mathbb{R}^{m+1}$ is imbedded in $\mathcal{A}$.

There are several ways to define a product in a Clifford algebra. In our theory the product in $\mathcal{A}$ is defined by the relations: $e_k^2 = -1$ and $e_k e_j + e_j e_k = 0$, whenever $k \neq j$, $k,j = 1,\dots,m$.

Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $f \in C_1(\Omega, \mathcal{A})$.

Then f is left (resp. right) monogenic in Ω if

$$D f = \sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j} f = 0 \quad (\text{resp. } f D = \sum_{j=0}^{m} \frac{\partial}{\partial x_j} f e_j = 0)$$

in Ω.

The right $\mathcal{A}$-module of left monogenic functions in Ω is denoted by $M_1(\Omega, \mathcal{A})$. It is a Fréchet module for the topology of uniform convergence on the compact subsets of Ω.

For the basic elementary function theoretic theorems we refer to (2).

For any open subset Ω of $\mathbb{C}^m$, $\mathcal{H}_{(1)}(\Omega, \mathcal{A})$ (resp. $\mathcal{H}_{(r)}(\Omega, \mathcal{A})$) is the left (resp. right) module of $\mathcal{A}$-valued holomorphic functions in Ω. Hence its dual module: $\mathcal{H}'_{(1)}(\Omega, \mathcal{A})$, consists of left linear $\mathcal{A}$-valued analytic functionals.

ω_{m+1} is the area of the unit sphere in $\mathbb{R}^{m+1}$.

I. Special monogenic functions.

Let $u = u_0 + \vec{u}$ belong to $\mathbb{R}^{m+1}$ let $\vec{z} = \vec{x} + i\vec{y} = \sum_{j=1}^{m} e_j z_j$ belong to $\mathbb{C}^m$ and put $<\vec{u}, \vec{z}> = \sum_{j=1}^{m} u_j z_j$.

Then one easily shows that the functions $(<\vec{u}, \vec{z}> - u_0 \vec{z})^k$, which are defined for $(u, \vec{z}) \in \mathbb{R}^{m+1} \times \mathbb{C}^m$, are left and right monogenic for $u \in \mathbb{R}^{m+1}$ and holomorphic for $\vec{z} \in \mathbb{C}^m$.

Furthermore, when f is a $\mathbb{C}$-valued holomorphic function in $\{z \in \mathbb{C} \,\|\, |z| < \rho\}$, $\rho > 0$, admitting the Taylor series expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$, then the function $f(u.z)$, $(u, z) \in \mathbb{C}^2$, can be generalized immediately to the function

$$(1) \quad F(u, \vec{z}) = \sum_{k=0}^{\infty} c_k (<\vec{u}, \vec{z}> - u_0 \vec{z})^k, \quad (u, \vec{z}) \in \mathbb{R}^{m+1} \times \mathbb{C}^m,$$

which is left and right monogenic in u and holomorphic in $\vec{z}$.

In the followimg theorem we study the convergence of the series $\sum_{k=0}^{\infty} c_k (<\vec{u}, \vec{z}> - u_0 \vec{z})^k$.

<u>Theorem I.</u> Let $f(z) = \sum_{k=0}^{\infty} c_k z^k$ be holomorphic in $\{z \in \mathbb{C} \,|\, |z| < \rho\}$, $\rho > 0$. Then the series $\sum_{k=0}^{\infty} c_k (<\vec{u}, \vec{z}> - u_0 \vec{z})^k$ converges absolutely as a multiple Taylor series of $u_0, \ldots, u_m$ in the domain

$$\left\{ u \in \mathbb{R}^{m+1} \,\|\, |u_0| \left(\sum_{j=1}^{m} |z_j|^2 \right)^{1/2} + \sum_{j=1}^{m} |u_j| \, |z_j| < \rho \right\}.$$

In the case $\vec{z} = \vec{x} \in \mathbb{R}^m$, this domain is optimal.

<u>Proof.</u> As f is holomorphic for $z \in \mathbb{C}$, $|z| < \rho$ and as

$$(<\vec{u}, \vec{z}> - u_0 \vec{z})^k = \sum_{\sum_{j=0}^{m} k_j = k} \frac{k!}{k_0! \ldots k_m!} (-u_0 \vec{z})^{k_0} \prod_{j=1}^{m} (u_j z_j)^{k_j},$$

the domain of absolute convergence of the multiple Taylor series $\sum_{k=0}^{\infty} c_k (<\vec{u}, \vec{z}> - u_0 \vec{z})^k$ is determined by the condition

$$\sum_{k=0}^{\infty} |c_k| \sum_{\sum_{j=0}^{m} k_j = k} \frac{k!}{k_0! \ldots k_m!} |u_0|^{k_0} |\vec{z}^{k_0}| \prod_{j=1}^{m} (|u_j| |z_j|)^{k_j} < \infty \quad (*).$$

For $k_o = 2s$, $s \in \mathbb{N}$, $\vec{z}^{\,k_o} = (-1)^s \left(\sum_{j=1}^{m} z_j^2\right)^s$

and for $k_o = 2s+1$, $s \in \mathbb{N}$, $\vec{z}^{\,k_o} = (-1)^s \left(\sum_{j=1}^{m} z_j^2\right)^s \vec{z}$.

Hence, for any $k_o \in \mathbb{N}$, $|\vec{z}^{\,k_o}| \leqslant \left(\sum_{j=1}^{m} |z_j|^2\right)^{k_o/2}$

and when $\vec{z} = \vec{x} \in \mathbb{R}^m$, $|\vec{x}^{\,k_o}| = \left(\sum_{j=1}^{m} x_j^2\right)^{k_o/2}$.

Hence (*) is satisfied as soon as $(u, \vec{z})$ satisfies the inequality

$$|u_o| \left(\sum_{j=1}^{m} |z_j|^2\right)^{1/2} + \sum_{j=1}^{m} |u_j||z_j| < \rho$$

and when $\vec{z} = \vec{x} \in \mathbb{R}^m$, (*) is equivalent with this inequality. ∎

We give two important examples.

Example I.

$$P(u, \vec{z}) = \sum_{k=0}^{\infty} (<\vec{u}, \vec{z}> - u_o \vec{z})^k = \frac{1 - <\vec{u}, \vec{z}> - u_o \vec{z}}{(1 - <\vec{u}, \vec{z}>)^2 + u_o^2 \sum_{j=1}^{m} z_j^2}$$

is the generalization of $(1 - uz)^{-1}$, $(u, z) \in \mathbb{C}^2$.

Obviously $P(u, \vec{z})$ is defined in the open domain

$\mathcal{U} = (\mathbb{R}^{m+1} \times \mathbb{C}^m) \setminus S$, where S is given by the equations

$$(1 - <\vec{u}, \vec{x}>)^2 + u_o^2 |\vec{x}|^2 = <\vec{u}, \vec{y}>^2 + u_o^2 |\vec{y}|^2$$
$$(1 - <\vec{u}, \vec{x}>) <\vec{u}, \vec{y}> = u_o^2 <\vec{x}, \vec{y}> .$$

Example 2.

$$\begin{aligned} Exp(u, \vec{z}) &= \sum_{k=0}^{\infty} \frac{1}{k!} (<\vec{u}, \vec{z}> - u_o \vec{z})^k \\ &= e^{<\vec{u}, \vec{z}>}\left(\cos\left(u_o \sqrt{\sum_{j=1}^{m} z_j^2}\right) - \frac{\vec{z}}{\sqrt{\sum_{j=1}^{m} z_j^2}} \sin\left(u_o \sqrt{\sum_{j=1}^{m} z_j^2}\right) \right). \end{aligned}$$

The function $Exp(u, \sum_{j=1}^{m} e_j)$ has been studied explicitly by Brackx in (1).

In (8) we introduced another way to generalize holomorphic functions. Let f be holomorphic in $\{z \in \mathbb{C} \| |z| < \rho\}$ and admit the Taylor expansion $f(z) = \sum_{k=0}^{\infty} c_k z^k$.
Then the function $\frac{1}{z} f(\frac{u}{z})$, $(u, z) \in \mathbb{C} \times (\mathbb{C} \setminus \{0\})$ is generalized to the function

$$(2)\quad F(u, y) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} c_k \left(\sum_{j=0}^{m} u_j \frac{\partial}{\partial y_j} \right)^k \frac{\bar{y}}{|y|^{m+1}} ,$$

which is defined for $(u, y) \in \mathbb{R}^{m+1} \times (\mathbb{R}^{m+1} \setminus \{0\})$ such that $|u| < \rho |y|$, and which is left and right monogenic in both variables u and y separately.
In the following theorem we give a relation between the above introduced generalizations of a holomorphic function.

Theorem 2. Let $f(z) = \sum_{k=0}^{\infty} c_k u^k$ be holomorphic for $|z| < \rho$ and let $F(u, \vec{z})$ and $F(u, y)$ be the generalizations of $f(u.z)$ and $\frac{1}{z} f(\frac{u}{z})$, introduced in (1) and (2) respectively. Then for sufficiently small $r > 0$ and $|u| < \rho r$,

$$F(u, \vec{z}) = \frac{1}{\omega_{m+1}} \int_{\partial B(0, r)} F(u, y)\, d\sigma_y\, P(y, \vec{z}) .$$

Proof. Let $|z_j| < R$; $j=1, \ldots, m$.
Then $P(u, \vec{z})$ is left monogenic in

$$A = \left\{u \in \mathbb{R}^{m+1} \| |u_0| \sqrt{m} + \sum_{j=1}^{m} |u_j| < R^{-1} \right\} .$$

Choose $r > 0$ such that $\bar{B}(0, r) = \left\{u \in \mathbb{R}^{m+1} \| |u| < r\right\} \subset A$.
Then $\frac{1}{\omega_{m+1}} \int_{\partial B(0, r)} F(u, y)\, d\sigma_y\, P(y, \vec{z})$

is defined for $|u| < \rho r$ and in view of (7) it admits a Taylor series expansion which is exactly equal to

$$\sum_{k=0}^{\infty} c_k (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k . \blacksquare$$

2. The Radon transform

In this section we study the monogenic version of the Radon transform.

Definition I. Let $\Omega \subset \mathbb{C}^m$ be a domain of holomorphy and let $T \in \mathcal{H}'_{(1)}(\Omega, \mathcal{A})$.

Then we define the Radon transform by

$$\Pi(T)(u) = \langle T_{\vec{z}}, P(u, \vec{z}) \rangle .$$

Definition 2. Let $\Omega \subset \mathbb{R}^{m+1}$ be open and let $T \in M'_1(\Omega, \mathcal{A})$.

Then we define the converse Radon transform by

$$\mu(T)(\vec{z}) = \langle T_u, P(u, \vec{z}) \rangle .$$

Observe that both Π and μ are generalizations of the classical Radon transform

$$\Pi(T)(z) = \langle T_u, (1 - u.z)^{-1} \rangle , \quad T \in \mathcal{H}'(\Omega),$$

and that $\Pi(T.a) = \Pi(T).a$ and $\mu(a.T) = a.\mu(T)$ for all $a \in \mathcal{A}$.

Π maps complex analytic functionals into left monogenic functions and μ maps analytic functionals in the monogenic sense into holomorphic functions.

In this paper we study the image of the transform Π in some special interesting cases.

Let $R_1 > 0, \ldots, R_m > 0$. Then we put:

a. $B(R_1, \ldots, R_m) = \{ \vec{z} \in \mathbb{C}^m \| \; |z_j| < R_j \}$

b. $P(R_1, \ldots, R_m) = \{ u \in \mathbb{R}^{m+1} \| \; \sum_{j=1}^m R_j |u_j| + \sqrt{\sum_{j=1}^m R_j^2}\, |u_o| < 1 \}$

c. $b(R_1, \ldots, R_m) = \{ \vec{z} \in \mathbb{C}^m \| \; \sum_{j=1}^m R_j |z_j| < 1 \}$

d. $p(R_1, \ldots, R_m) = p_1(R_1) \cap \ldots \cap p_m(R_m)$

where $p_j(R_j) = \{ u \in \mathbb{R}^{m+1} \| \; |u_j| + |u_o| < R_j \}$.

Furthermore we put

$$\mathcal{H}_{(1)}(\bar{B}(R_1, \ldots, R_m)) = \lim_{\varepsilon > 0} \operatorname{ind} \mathcal{H}_{(1)}(B(R_1 + \varepsilon, \ldots, R_m + \varepsilon), \mathcal{A})$$

and

$$\mathcal{H}_{(1)}(\bar{b}(R_1, \ldots, R_m)) = \lim_{\varepsilon > 0} \operatorname{ind} \mathcal{H}_{(1)}(b(R_1 + \varepsilon, \ldots, R_m + \varepsilon), \mathcal{A})$$

In Theorem 3 and Theorem 4 we give a characterization of $\pi(\mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m)))$.

Theorem 3. Let $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))$.
Then $\pi(T)(u)$ is left monogenic in $P(R_1, \ldots, R_m)$ and its multiple Taylor series converges absolutely in $P(R_1, \ldots, R_m)$. Furthermore $P(R_1, \ldots, R_m)$ is optimal for the absolute convergence of multiple Taylor series of monogenic functions.

Proof. Let $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))$. Then for each $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|\langle T, \varphi \rangle| \leqslant C_\varepsilon \sup_{\vec{z} \in B(R_1, \ldots, R_m)} |\varphi(\vec{z})|$$

for every $\varphi \in \mathcal{H}_{(1)}(\bar{B}(R_1 + 2\varepsilon, \ldots, R_m + 2\varepsilon))$.
Let $u \in P(R_1, \ldots, R_m)$ be fixed and choose $\varepsilon > 0$ such that $u \in P(R_1 + 2\varepsilon, \ldots, R_m + 2\varepsilon)$. Then for fixed u , $P(u, \vec{z}) \in \mathcal{H}_{(1)}(\bar{B}(R_1 + 2\varepsilon, \ldots, R_m + 2\varepsilon))$ and hence, $\pi(T)(u)$ is defimed in $P(R_1, \ldots, R_m)$.
Furthermore,

$$\langle T_{\vec{z}} , (\langle \vec{u}, \vec{z} \rangle - u_0 \vec{z})^k \rangle$$

$$= \sum_{\sum_{j=0}^{m} k_j} \frac{k!}{k_0! \ldots k_m!} u_0^{k_0} \ldots u_m^{k_m} \langle T_{\vec{z}} , (-\vec{z})^{k_0} \prod_{j=1}^{m} z_j^{k_j} \rangle$$

and

$$|\langle T_{\vec{z}}\, ,\, (-\vec{z}\,)^{k_o} \prod_{j=1}^{m} z_j^{\,k_j} \rangle|$$

$$\leq C_\varepsilon \sup_{\bar{B}(R_1+\varepsilon,\ldots,R_m+\varepsilon)} |\vec{z}\,|^{k_o} \prod_{j=1}^{m} |z_j|^{k_j}$$

$$\leq C_\varepsilon \Big(\sum_{j=1}^{m} (R_j + \varepsilon)^2 \Big)^{k_o/2} \prod_{j=1}^{m} (R_j + \varepsilon)^{k_j}$$

Hence,

$$\sum_{\sum_{j=1}^{m} k_j = k} \frac{k!}{k_o!\ldots k_m!} |u_o|^{k_o} \ldots |u_m|^{k_m} |\langle T_{\vec{z}}\, ,\, (-\vec{z}\,)^{k_o} \prod_{j=1}^{m} z_j^{\,k_j} \rangle|$$

$$\leq C_\varepsilon \Big(\sum_{j=1}^{m} |u_j|(R_j + \varepsilon) + |u_o| \Big(\sum_{j=1}^{m} (R_j + \varepsilon)^2 \Big)^{1/2} \Big)^k$$

$\leq C_\varepsilon (1 - \delta_\varepsilon)^k$, for some $\delta_\varepsilon > 0$; which implies that the multiple Taylor series of $\Pi(T)(u)$ converges absolutely in $P(R_1, \ldots, R_m)$.

Furthermore, $\delta_{(R_1,\ldots,R_m)} \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))$ and

$$\Pi(\delta_{(R_1,\ldots,R_m)})(u) = \frac{1 - \sum_{j=1}^{m} R_j u_j - u_o \sum_{j=1}^{m} R_j e_j}{(1 - \sum_{j=1}^{m} R_j u_j)^2 + u_o^2 \sum_{j=1}^{m} R_j^2}$$

admits a multiple Taylor series, which, in view of Theorem I, converges absolutely in $P(R_1, \ldots, R_m)$, but not in any point outside of $P(R_1, \ldots, R_m)$. This means that $P(R_1, \ldots, R_m)$ is optimal for the absolute convergence of multiple Taylor series of monogenic functions. ∎

<u>Theorem 4.</u> Let f be left monogenic in $P(R_1, \ldots, R_m)$ such that the multiple Taylor series of f converges absolutely in $P(R_1, \ldots, R_m)$. Then $f = \Pi(T)$ for some $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))$.

Proof. In view of (4), one easily shows that the mapping

$$\tilde{\Pi} : T_{\vec{z}} \longrightarrow \langle T_{\vec{z}} , (1 - \sum_{j=1}^{m} \zeta_j z_j)^{-1} \rangle = \tilde{\Pi}(T)(\vec{\zeta})$$

is a topological isomorphism between $\mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))_{,b}$ and $\mathcal{H}_{(r)}(b(R_1, \ldots, R_m))$. (and also between $\mathcal{H}'_{(1)}(\bar{b}(R_1, \ldots, R_m))_{,b}$ and $\mathcal{H}_{(r)}(B(R_1, \ldots, R_m))$.)

On the other hand, $f|_{u_0=0}$ admits a multiple Taylor series expansion:

$$f|_{u_0=0}(u_1, \ldots, u_m) = \sum_{k=0}^{\infty} \sum_{\sum_{j=1}^{m} k_j = k} u_1^{k_1} \ldots u_m^{k_m} a_{k_1,\ldots,k_m}$$

which converges absolutely in $\{ \vec{u} \in \mathbb{R}^m \| \sum_{j=1}^{m} |u_j| R_j < 1 \}$.

Hence, $f(\zeta_1, \ldots, \zeta_m) = \sum_{k=0}^{\infty} \sum_{\sum_{j=1}^{m} k_j = k} \zeta_1^{k_1} \ldots \zeta_m^{k_m} a_{k_1,\ldots,k_m}$,

which is the holomorphic extension of $f|_{u_0=0}(u_1, \ldots, u_m)$, belongs to $\mathcal{H}_{(r)}(b(R_1, \ldots, R_m))$.

Hence $f|_{u_0=0}(u_1, \ldots, u_m) = \langle T_{\vec{z}} , P(u, \vec{z}) \rangle|_{u_0=0}$, for some $T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))$.

As analytic functions in open subsets of $\mathbb{R}^m$ admit unique left monogenic extensions to open subsets of $\mathbb{R}^{m+1}$ (Theorem 6.), we obtain that $f(u) = \langle T_{\vec{z}} , P(u, \vec{z}) \rangle$. ∎

In view of Theorem 3. and Theorem 4 , $\Pi(\mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m)))$ coincides with the right module of left monogenic functions in $P(R_1, \ldots, R_m)$ of which the multiple Taylor series converges absolutely in $P(R_1, \ldots, R_m)$.

In an analogous way one shows:

Theorem 5. $\pi(\mathcal{H}'_{(1)}(\mathfrak{b}(R_1, \ldots, R_m)))$ coincides with the right module of left monogenic functions in $p(R_1, \ldots, R_m)$ of which the multiple Taylor series converges absolutely in $p(R_1, \ldots, R_m)$.

3. The Cauchy-Kowalewski extension theorem .

Let $\Omega \subset \mathbb{R}^m$ be open. Then an open subset $\tilde{\Omega}$ of $\mathbb{R}^{m+1}$ is called a normal open neighbourhood of Ω in $\mathbb{R}^{m+1}$ when for each point $u \in \tilde{\Omega}$, $u = u_0 + \vec{u}$, the set $\{x \in \mathbb{R}^{m+1} \| x = x_0 + \vec{u} \text{ and } x \in \tilde{\Omega}\}$ is connected and contains one point of Ω .

In (6) we showed the following Cauchy-Kowalewski type extension theorem.

Theorem 6. Let $\Omega \subset \mathbb{R}^m$ be open and let f be an $\mathcal{A}$ - valued analytic function in Ω . Then there exists an normal open neighbourhood $\tilde{\Omega}$ of Ω in $\mathbb{R}^{m+1}$ and a left monogenic function f' in $\tilde{\Omega}$ such that $f'(\vec{x} + x_0)|_{x_0=0} = f(\vec{x})$. Furthermore, if f'_1 and f'_2 are left monogenic extensions of f in open normal neighbourhoods $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$ of Ω, then $\tilde{\Omega}_1 \cap \tilde{\Omega}_2$ is a normal open neighbourhood of Ω and $f'_1|_{\tilde{\Omega}_1\cap\tilde{\Omega}_2} = f'_2|_{\tilde{\Omega}_1\cap\tilde{\Omega}_2}$. Hence there exists a unique left monogenic extension of f which is defined in a maximal open and normal neighbourhood of Ω .

In the following theorem we give a characterization of the multiple Taylor series convergence of the Cauchy-Kowalewski extensions of a special class of analytic functions.

Theorem 7. Let $f(z_1, \ldots, z_m)$ be an $\mathcal{A}$- valued holomorphic function in $B(R_1, \ldots, R_m)$ and let $f(x_1, \ldots, x_m)$ be its restriction to $\mathbb{R}^m$. Then $f(x_1, \ldots, x_m)$ admits a unique left monogenic extension $f(x_0 + \vec{x})$ in $p(R_1, \ldots, R_m)$, which admits an absolutely converging multiple Taylor series expansion in $p(R_1, \ldots, R_m)$ and which is given by

$$f(x_0 + \vec{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x_0^{2k} \Delta_m^k f(x_1, \ldots, x_m)$$
$$- \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x_0^{2k+1} \left(\sum_{j=1}^{m} e_j \frac{\partial}{\partial x_j}\right) \Delta_m^k f(x_1, \ldots, x_m).$$

Furthermore $p(R_1, \ldots, R_m)$ is optimal.

Proof. As $f(z_1, \ldots, z_m) \in \mathcal{H}_{(r)}(B(R_1, \ldots, R_m))$,
$f(z_1, \ldots, z_m) = \langle T_{\vec{\zeta}}, (1 - \sum_{j=1}^{m} \zeta_j z_j)^{-1} \rangle$, for some
$T \in \mathcal{H}'_{(1)}(\bar{B}(R_1, \ldots, R_m))$.
Hence by Theorem 5, $f(x_1, \ldots, x_m) = \langle T_{\vec{\zeta}}, P(x, \vec{\zeta}) \rangle \big|_{x_0=0}$ admits the left monogenic extension:
$f(x_0 + \vec{x}) = \langle T_{\vec{\zeta}}, P(x_0 + \vec{x}, \vec{\zeta}) \rangle$ in $p(R_1, \ldots, R_m)$, which admits an absolutely converging multiple Taylor series expansion in $p(R_1, \ldots, R_m)$.
On the other hand, one can easily show that

$$f'(x_0 + \vec{x}) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x_0^{2k} \Delta_m^k f(x_1, \ldots, x_m)$$
$$- \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x_0^{2k+1} \left(\sum_{j=1}^{m} e_j \frac{\partial}{\partial x_j}\right) \Delta_m^k f(x_1, \ldots, x_m)$$

is left monogenic in $p(R_1, \ldots, R_m)$ and that
$f(x_1, \ldots, x_m) = f'(x_0 + \vec{x}) \big|_{x_0=0}$.

Hence by Theorem 6, $f = f'$.
We now show that $p(R_1, \ldots, R_m)$ is optimal.
Let $u \notin p(R_1, \ldots, R_m)$. Then for some $j = 1, \ldots, m$; $u \notin p_j(R_j)$. One can easily show that the function $f(x) = (R_j - (x_j - x_o e_j))^{-1}$ is left monogenic in $p(R_1, \ldots, R_m)$ and that its multiple Taylor series, which is given by $R_j^{-1} \sum_{k=o}^{\infty} (R_j^{-1}(x_j - x_o e_j))^k$,

converges absolutely for any $x \in p(R_1, \ldots, R_m)$, but not for $x = u$. ∎

The author wrote this paper during his stay at Sophia University in Tokyo, Japan.
He whishes to thank the Department of Mathematics of Sophia University for their support during his stay.
He whishes to thank Professor M. Morimoto for his constant interest in the theory of monogenic functions and for the interesting discussions about this and related topics.

References.

I. F. Brackx, The exponential function of a quaternion variable, Applicable Analysis 8 (1979) 265-276.

2. R. Delanghe, F. Brackx, Hypercomplex function theory and Hilbert modules with reproducing kernel, Proc. London Math. Soc. 37 (1978) 545-576.

3. W. K. Hayman, Power series expansions for harmonic functions, Bull. London Math. Soc. 2 (I970) I52-I58.

4. A. Martineau, Equations différentielles d'ordre infini, Bull. Soc. Math. France 95 (I967) I09-I54.

5. J. Siciak, Holomorphic continuation of harmonic functions, Ann. Pol. Math. 29 (I974) 67-73.

6. F. Sommen, A product and an exponential function in hypercomplex function theory, to appear in Applicable Analysis.

7. __________, Spherical monogenic functions and analytic functionals on the unit sphere, to appear.

8. __________, A generalized version of the Fourier-Borel transform, to appear.

Seminar of Higher Analysis
State University of Ghent
Krijgslaan 27I
B-9000 Gent
Belgium.